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On the radiation and scattering of short surface waves. Part 3

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The radiation properties of partially immersed three-dimensional bodies, in time-periodic motion, are examined in the short-wave asymptotic limit $\epsilon \to 0$, where ϵ is a non-dimensional wavelength. The method of matched expansions is used to specify an outer approximation, away from the surface wave region, and an inner approximation where the potential, in the vicinity of the obstacle and free surface, depends only on the local geometry. Finally, the radially spreading surface wave field is estimated by ray-theory arguments. Explicit details are given for the heaving and rolling of a circular dock and for the heaving motion of a hemisphere. Some speculations are made regarding the scattering properties of such obstacles.

1. Introduction

A method of matched expansions has been used in earlier work (Leppington 1972, 1973) to deal with problems of radiating and scattering of short surface waves by two-dimensional obstacles. The aim of this paper is to generalize these ideas in order to estimate the radiation due to a three-dimensional body that undergoes time-periodic oscillations.

If S denotes the surface of such a body and C is the (convex) curve where S intersects the free surface, co-ordinates (x, y, z) are chosen so that the fluid occupies the region $z > 0$, outside S. For time-periodic motions the linearized problem for the velocity potential $\mathscr{R}\{\phi(\mathbf{x}) \exp(-i\omega t)\}$ is

$$\phi_{xx}+\phi_{yy}+\phi_{zz} = 0 \quad \text{outside } S, \tag{1.1}$$

$$\partial\phi/\partial n = V \quad \text{on } S, \tag{1.2}$$

$$\phi+\epsilon\phi_z = 0 \quad \text{outside } C, \quad z = 0, \tag{1.3}$$

where suffixes denote partial differentiation, $\mathbf{n}$ is the outward normal from S, $V\exp(-i\omega t)$ is the prescribed normal velocity on S, and $\epsilon = g/\omega^2$ is a wavelength parameter. In addition, we require an outgoing wave condition

$$\phi \sim A(\lambda)\, r^{-\frac{1}{2}} \exp\{(ir-z)/\epsilon\} \quad \text{as} \quad r/a \to \infty, \tag{1.4}$$

where a is the maximum diameter of C, (r, λ, z) are cylindrical polars and $A(\lambda)$ is an amplitude parameter to be found. Our aim is to find an asymptotic approximation for ϕ and particularly for $A(\lambda)$ in the short-wave limit $\epsilon/a \to 0$.

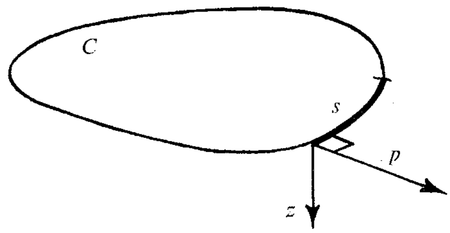

FIGURE 1. The obstacle S meets the free surface $z = 0$ at the curve C. The local co-ordinates X and Y are related to p and z by the formulae $p = \epsilon X$ and $z = \epsilon Y$.

The procedure for finding an asymptotic solution for ϕ follows closely that of parts 1 and 2 of this work (Leppington 1972, 1973). An *outer* approximation ϕ_0 is specified by taking $\epsilon = 0$ in condition (1.3) to get $\phi_0 = 0$ on the free surface, and by requiring $\phi_0 \to 0$ at infinity.

For points close to C, on the other hand, the proper surface condition (1.3) must be applied, but the solution is assumed to depend only on the *local* geometry of S, thus suggesting a rescaling of co-ordinates with respect to wavelength. Specifically, if s denotes arc length along C measured from some arbitrary starting-point, and p denotes the shortest distance from vertical generators through C, we define new local co-ordinates X and Y by the relations

$$p = \epsilon X, \quad z = \epsilon Y, \quad \phi(\mathbf{x}) = \Phi(s; X, Y) \tag{1.5}$$

to determine the local field near the point s (see figure 1).

This leads to a two-dimensional problem for the potential $\Phi \sim \alpha(\epsilon)\,\Phi_0$ near s. The specifications for the outer and inner approximations are completed by requiring them to match in their common region of validity. Details are provided in §§ 2, 3 and 4 for some special cases in which C is a circle, when the approximating potentials ϕ_0 and Φ_0 can be calculated explicitly.

In particular, we find the amplitude of the locally two-dimensional waves,

$$\alpha\Phi_0 \sim A_0(s;\epsilon)\exp(iX - Y) \quad \text{as} \quad X = p/\epsilon \to \infty, \tag{1.6}$$

that are formed, together with wave-free terms, near the point s. The amplitude parameter A_0 can be calculated, in principle, to any order of accuracy and is a slowly varying function of arc length s, for the radiation problems discussed here.

Now in the corresponding two-dimensional problems, the wave train (1.6) was simply extended from the inner region to the whole of the remaining free surface. For the three-dimensional geometries envisaged here, we have the additional task of calculating the way in which the locally two-dimensional waves (1.6), in the inner region, change into the radially spreading waves (1.4).

It is now proposed that this problem can be handled by standard *ray-theory* arguments. For if we use ϕ_w to denote the surface wave part of the potential, to the order of approximation contained in the estimate (1.6), then formulae (1.6) and (1.4) require that

$$\phi_w \sim \begin{cases} A_0(s;\epsilon)\exp\{(ip - z)/\epsilon\} & (\epsilon \ll p \ll a), \quad (1.7) \\ Ar^{-\frac{1}{2}}\exp\{(ir - z)/\epsilon\} & (r \gg a). \quad (1.8) \end{cases}$$

Although (1.7) is valid at the outer extremity of the inner region, the distance p is very small compared with the obstacle dimension a, so that in terms of the outer co-ordinates (r, z) formula (1.7) serves as a boundary condition on C. Now each of the two expressions (1.7) and (1.8) contain the same z dependence, and this suggests that we write

$$\phi_w = F(x, y; \epsilon) \exp(-z/\epsilon), \tag{1.9}$$

which ensures that ϕ_w satisfies the free-surface condition (1.3) identically. The problem for F is therefore

$$\left.\begin{aligned} F_{xx}+F_{yy}+\epsilon^2 F &= 0 \quad \text{outside } C, \\ F &= A_0(s;\epsilon) \quad \text{on } C, \\ F &\sim Ar^{-\frac{1}{2}}\exp(ir/\epsilon) \quad \text{as} \quad r\to\infty, \end{aligned}\right\} \tag{1.10}$$

which is a familiar radiation problem in acoustics and electromagnetic wave theory. Its asymptotic solution is found by the ray-theory method (Keller, Lewis & Seckler 1956), which is to write

$$F \sim \sum_0^\infty \epsilon^n F_n(x, y) \exp(ip/\epsilon), \tag{1.11}$$

substitute into (1.10) and equate equal powers of ϵ. Retaining only the leading term of the solution, we find that

$$F \sim (1+p/R)^{-\frac{1}{2}} A_0(s;\epsilon) \exp(ip/\epsilon) \quad \text{as} \quad \epsilon\to 0, \tag{1.12}$$

where $R = R(s)$ is the local radius of curvature at s, which is the point of C closest to the field point. Higher order terms can be written down to any order of accuracy, if required.

In the particular problems studied in §§ 2, 3 and 4, C is a circle of unit radius, so that $R = 1$ and $p = r-1$; thus

$$\phi_w \sim A_0(s)\, r^{-\frac{1}{2}} \exp\{i(r-1)/\epsilon - z/\epsilon\}. \tag{1.13}$$

To the first order of approximation then, the distant radial field (1.4) is given from the local inner field (1.6) by simply multiplying by $r^{-\frac{1}{2}}$. (I am grateful to a referee for pointing out this fact, which has led to the more general treatment given here.) If the inner wave field (1.6) were calculated to higher order, to give a more accurate estimate for $A_0(s)$, then the higher order terms of expansion (1.11) would have to be included to the same order of accuracy.

Sections 2 and 3 contain details of the solution of the problem of a circular dock that undergoes a heaving or rolling motion, and the analysis for the problem of a heaving hemisphere is given in § 4.

Numerical results have been obtained for these problems by MacCamy (1961) and by Kim (1963*a*, *b*), these being essentially long-wave computations with ϵ ranging from ∞ to $\frac{1}{4}$. A comparison for the smallest value $\epsilon = \frac{1}{4}$ shows that the asymptotic estimates (2.22), (3.9) and (4.23) below for the amplitude constants are in error by about 50 %. Since this value of ϵ corresponds to waves of length $1\frac{1}{2}$ times the radius, it is not small enough for short-wave asymptotics to be accurate.

2. Radiation by a heaving circular dock

A circular dock of unit radius and zero thickness lies on the otherwise-free surface of a fluid of great depth, and undergoes a small amplitude heaving motion that is simple harmonic in time. Cylindrical polar co-ordinates (r, λ, z) are chosen so that the dock occupies the region $r < 1$, $z = 0$, with fluid in the half-space $z > 0$. The velocity potential is independent of λ and has the form $\mathscr{R}\{\phi(r,z)\,e^{-i\omega t}\}$, where ω is the angular frequency; the time factor $e^{-i\omega t}$ will henceforth be suppressed.

For small vibrations, with prescribed downward velocity $\mathscr{R}\{V_0 e^{-i\omega t}\}$ for the dock, the potential ϕ is specified by the linearized boundary-value problem

$$\phi_{rr} + r^{-1}\phi_r + \phi_{zz} = 0 \quad (z > 0), \tag{2.1}$$

with

$$\phi_z = V_0 \quad (z = 0, r < 1), \tag{2.2}$$

and

$$\phi + \epsilon\phi_z = 0 \quad (z = 0, r > 1). \tag{2.3}$$

Here V_0 is a constant, and $\epsilon = g/\omega^2$ is $1/2\pi$ times the ratio of wavelength to dock radius. There is also an edge condition

$$|\nabla\phi| = O(\delta^k) \quad \text{for some} \quad k > -1 \quad \text{as} \quad \delta \to 0, \tag{2.4}$$

where δ is the distance from the edge of the dock. Finally an outgoing wave condition is imposed, thus

$$\phi \sim A r^{-\frac{1}{2}} \exp\{(ir - z)/\epsilon\} \quad \text{as} \quad r \to \infty, \tag{2.5}$$

where A is to be found in the limit of small ϵ.

The procedure for finding an asymptotic estimate for ϕ throughout most of the fluid region is very similar to that described in parts 1 and 2 for the related two-dimensional problem. An 'outer' approximation ϕ_0, presumed valid at distances of many wavelengths from the free surface, is specified by formally setting $\epsilon = 0$ in the boundary condition (2.3). By adding a wave train of the form (1.13), with A_0 suitably chosen, this outer approximation has its region of validity extended to cover all points at distance $\gg \epsilon$ from the rim $r = 1$, $z = 0$.

A different approximation is posed for points that are near, on the dock-radius scale, to the edge. For in this vicinity the potential must satisfy the correct boundary condition (2.3), but will be aware only of the local geometry. This suggests rescaling co-ordinates with respect to wavelength, and leads to an 'inner' potential problem that involves a dock of *semi-infinite* extent. The rescaling and the formal procedure, due to Van Dyke (1964, ch. 4, 5), for matching the two different approximations in their common region of validity are described in detail below.

Outer approximation

Dealing first with the 'outer' approximation ϕ_0, we have to solve Laplace's equation, with boundary conditions on $z = 0$

$$\phi_{0z} = V_0 \quad (r < 1), \qquad \phi_0 = 0 \quad (r > 1). \tag{2.6}$$

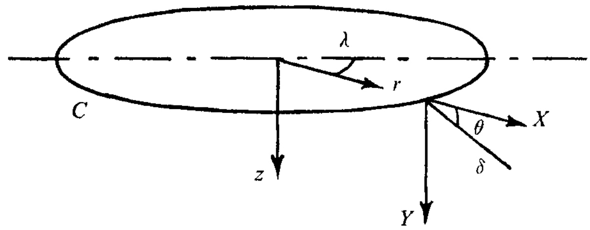

FIGURE 2. The cylindrical polar co-ordinates (r, λ, z) and the local co-ordinates (X, Y) and (δ, θ). The inner and outer variables R and δ are related by the formula $\delta = \epsilon R$, and C is the curve where the obstacle intersects the fluid.

Such an axisymmetric potential problem is readily handled by the methods described by Sneddon (1966, ch. 4), and one can verify that a solution is given by

$$\phi_0(r, z) = \frac{2V_0}{\pi}\int_0^\infty \frac{d}{ds}\left(\frac{\sin s}{s}\right) e^{-sz} J_0(sr)\, ds, \tag{2.7}$$

where J_0 denotes a Bessel function. The solution (2.7) is the one that satisfies the edge condition (2.4). Since the edge $(r = 1, z = 0)$ lies outside the region of validity of ϕ_0 we are not entitled to assume, without further justification, that such an edge condition must hold there, and could add any eigensolution of the problem. As in the two-dimensional case, such eigenfunctions behave like $\delta^{-\frac{1}{2}}$, or worse, near an edge and are rejected on the grounds that they cannot be matched with any inner solution.

In order to examine the form of the 'inner' solution near the rim, we need to determine the behaviour of ϕ_0 near $r = 1, z = 0$. It is convenient to calculate first the value of ϕ_0 on the dock, with $z = 0$ and $r < 1$. A partial integration of the integral (2.7) shows that

$$\begin{aligned}\phi_0(r, 0) &= -(2V_0/\pi) + (2V_0/\pi)\, r\int_0^\infty \frac{\sin s}{s} J_1(sr)\, ds \\ &= -(2V_0/\pi)(1-r^2)^{\frac{1}{2}}, \quad \text{when} \quad r < 1,\end{aligned} \tag{2.8}$$

according to Watson (1944, p. 405). In particular, for points at a small distance δ from the edge, we set $r = 1-\delta$ and expand for small δ to get

$$\phi_0 \sim -(2V_0/\pi)(2^{\frac{1}{2}}\delta^{\frac{1}{2}} - 2^{-\frac{3}{2}}\delta^{\frac{3}{2}} + \ldots). \tag{2.9}$$

More generally, the nature of the solution at all points within a neighbourhood of the edge is found from (2.9) together with (2.1) and (2.6) to have the form

$$\phi_0 \sim -(2V_0/\pi)(2^{\frac{1}{2}}\delta^{\frac{1}{2}} \sin \tfrac{1}{2}\theta - \tfrac{1}{2}\pi\delta \sin\theta) + O(\delta^{\frac{3}{2}}), \tag{2.10}$$

in the local co-ordinate system (δ, θ) of figure 2.

Inner approximation

Formula (2.10) leads the way to our investigation of the potential near the edge, where ϕ_0 is not valid. In such a region, well within a dock radius from the edge, the potential is expected to vary on a wavelength scale; thus we change variables according to the transformation

$$r = 1 + \epsilon X, \quad z = \epsilon Y, \quad \phi(r, z) = \Phi(X, Y). \tag{2.11}$$

In the terms of the 'inner' variables (X, Y), with $R = (X^2+Y^2)^{\frac{1}{2}}$, the leading term of the expansion (2.10) takes the form

$$\phi_0 \sim -(V_0/\pi)\, 2^{\frac{3}{2}} \epsilon^{\frac{1}{2}} R^{\frac{1}{2}} \sin \tfrac{1}{2}\theta, \tag{2.12}$$

which suggests that

$$\phi \sim \epsilon^{\frac{1}{2}} \Phi_0(X, Y) \tag{2.13}$$

in the edge region. On substituting into the governing equations (2.1)–(2.5) and taking the limit $\epsilon \to 0$ it is found that $\Phi_0(X, Y)$ is the harmonic function that satisfies the surface condition

$$\Phi_0 + \Phi_{0Y} = 0, \tag{2.14}$$

together with an edge condition at $(X, Y) = (0, 0)$ and the matching condition (from (2.12)) that

$$\Phi_0 \sim -(V_0/\pi)\, 2^{\frac{3}{2}} R^{\frac{1}{2}} \sin \tfrac{1}{2}\theta \quad \text{as} \quad R \to \infty, \tag{2.15}$$

plus a regular wave train.

Apart from a proportionality constant, this potential is the same as the one that occurs in the corresponding two-dimensional problem (see part 1), and has the solution given by Holford (1964) as

$$\Phi_0 = -(V_0/\pi)\, 2^{\frac{3}{2}} \left\{ R^{\frac{1}{2}} \sin \tfrac{1}{2}\theta - \tfrac{1}{2}\pi^{-\frac{1}{2}} \int_0^\infty s^{-\frac{1}{2}} \Theta(X, Y; s)\, ds \right\}, \tag{2.16}$$

where

$$\Theta = \left\{ \begin{array}{ll} -i2^{\frac{1}{2}} \dfrac{\Lambda(s)}{1+s^2} \exp\{i(X - \tfrac{1}{8}\pi) - Y\} + \dfrac{\cos sY + s \sin sY}{1+s^2} e^{-sX} & \\ \qquad + \pi^{-1} \dfrac{\Lambda(s)}{s-i} ⨍_0^\infty \dfrac{(t \cos tY - \sin tY)\, e^{-tX}}{(t+i)\,\Lambda(t)} \dfrac{dt}{t-s} & (X > 0), \\ -\pi^{-1} \dfrac{\Lambda(s)}{s-i} \displaystyle\int_0^\infty \dfrac{\Lambda(t) \cos tY\, e^{tX}}{t(t-i)} \dfrac{dt}{t+s} & (X < 0). \end{array} \right\} \tag{2.17}$$

The crossed integral sign denotes a Cauchy principal value, and Λ is defined as

$$\Lambda(t) = t^{\frac{1}{2}} (1+t^2)^{\frac{1}{4}} \exp\left\{ -\frac{1}{\pi} \int_0^t \frac{\log u}{1+u^2} du \right\}. \tag{2.18}$$

The behaviour of Φ_0 for large positive values of X can be deduced from (2.16) and (2.17), and is found (see part 1) to be

$$\Phi_0 + (V_0/\pi)\, 2^{\frac{3}{2}} R^{\frac{1}{2}} \sin \tfrac{1}{2}\theta \sim -2i\pi^{-\frac{1}{2}} V_0\, e^{-\frac{1}{8}i\pi}\, e^{iX-Y}. \tag{2.19}$$

Wave train at infinity

According to formula (2.19), the inner approximation $\phi \sim \epsilon^{\frac{1}{2}} \Phi_0$, which is valid for $r - 1 \ll 1$, has the form

$$\Phi \sim \epsilon^{\frac{1}{2}} \Phi_0 \sim -2i(\epsilon/\pi)^{\frac{1}{2}} V_0\, e^{-\frac{1}{8}\pi i} \exp\{[i(r-1) - z]/\epsilon\} \tag{2.20}$$

as $(r-1)/\epsilon \to \infty$, together with a wave-free term. This is restricted to values of r such that $r - 1 \ll 1$, and represents the potential at the outer extremities of the inner region, where the outgoing waves are locally two-dimensional. A comparison with the formula (1.13) shows, however, that the outgoing wave train has the more general form

$$\Phi \sim A r^{-\frac{1}{2}} \exp\{(ir - z)/\epsilon\} \quad \text{as} \quad r \to \infty, \tag{2.21}$$

where

$$A \sim -2i(\epsilon/\pi)^{\frac{1}{2}} V_0 \exp(-\tfrac{1}{8}\pi i - i/\epsilon) \quad \text{as} \quad \epsilon \to 0, \tag{2.22}$$

to ensure consistency with (1.13). Thus the outer approximation is extended, to cover all points many wavelengths from the rim, by superimposing ϕ_0 on the wave train (2.21) with (2.22). In this way, both the wave-free terms and the outgoing wave trains are seen to match at the common points of the outer and inner regions, and the first-order approximation is complete.

3. Radiation by a rolling circular dock

An example of a three-dimensional problem which is not axially symmetric is that of the rolling circular dock, with the prescribed normal velocity $V(x) = V_1 x$. The analysis is similar to that of § 2 and will therefore be treated briefly. In terms of the cylindrical polars (r, λ, z) the fluid occupies the half-space $z > 0$, the dock is given by $r < 1$, $z = 0$, and the normal velocity is $V = V_1 r \cos\lambda$.

The outer potential ϕ_0 is now the harmonic function that vanishes at infinity and satisfies an edge condition and the boundary conditions

$$\phi_0(r, 0) = 0 \quad (r > 1); \qquad \phi_{0z}(r, 0) = V_1 r \cos\lambda \quad (r < 1). \tag{3.1}$$

This is again a problem of the type treated by Sneddon (1966), and has the solution

$$\phi_0 = -\frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} V_1 \cos\lambda \int_0^\infty s^{-\frac{1}{2}} J_{\frac{3}{2}}(s)\, J_1(sr)\, e^{-sz}\, ds, \tag{3.2}$$

where J_i denotes a Bessel function. It can be readily verified (from Watson 1944, p. 404) that the integral (3.2) satisfies the conditions (3.1), and that ϕ_0 has the value

$$\phi_0(r, 0) = -(4V_1/3\pi)\, r(1 - r^2)^{\frac{1}{2}} \cos\lambda \quad (r < 1) \tag{3.3}$$

on the dock. In particular, for points close to the rim we have

$$\phi_0(1 - \delta, 0) \sim -(4V_1/3\pi)\, 2^{\frac{1}{2}} \delta^{\frac{1}{2}} \cos\lambda \quad \text{as} \quad \delta \to 0,$$

whence

$$\phi_0 \sim -(4V_1/3\pi)\, 2^{\frac{1}{2}} \cos\lambda\, \delta^{\frac{1}{2}} \sin\tfrac{1}{2}\theta \quad \text{as} \quad \delta \to 0, \tag{3.4}$$

in the local co-ordinate system of figure 2.

On rewriting (3.4) in terms of the inner variable $R = \delta/\epsilon$, this formula suggests an inner approximation

$$\phi \sim \epsilon^{\frac{1}{2}} \Phi_0, \tag{3.5}$$

with

$$\Phi_0 \sim -(4V_1/3\pi)\, 2^{\frac{1}{2}} \cos\lambda\, R^{\frac{1}{2}} \sin\tfrac{1}{2}\theta \quad \text{as} \quad R \to \infty. \tag{3.6}$$

The potential Φ_0 is seen to be proportional to the corresponding function that occurs in § 2 for the heaving dock. In particular, its far-field form is given by

$$\Phi_0 \sim -\tfrac{4}{3} V_1 \pi^{-\frac{1}{2}} i \cos\lambda \exp(-\tfrac{1}{8}\pi i + iX - Y) \quad \text{as} \quad X \to \infty, \tag{3.7}$$

together with the wave-free term (3.6). A comparison of this result for the potential, $\phi \sim \epsilon^{\frac{1}{2}}\Phi_0$, with the general form (1.13) requires that

$$\phi \sim A_1 r^{-\frac{1}{2}} \cos\lambda \exp\{(ir-z)/\epsilon\} \quad \text{as} \quad r \to \infty, \tag{3.8}$$

where
$$A_1 \sim -i(\epsilon/\pi)^{\frac{1}{2}} \tfrac{4}{3} V_1 \exp(-\tfrac{1}{8}\pi i - i/\epsilon). \tag{3.9}$$

Since the problems of §§ 2 and 3 are linear, a simple superposition provides the solution for the case $V(x) = V_0 + xV_1$ of a combined heaving and rolling motion. The present method is obviously capable of extension to deal with any prescribed velocity $V(r, \lambda)$, by writing V as a Fourier series in λ and treating each component separately. The outer problems are classical mixed boundary-value problems of the type treated by Sneddon (1966), and the inner potentials will be proportional to Φ_0.

4. Radiation by a heaving hemisphere

The problem of calculating the short waves radiated by a heaving hemisphere can be attacked by a similar analysis. Since this obstacle is locally vertical near its intersection with the free surface, the inner potential is found to reduce to a problem of waves produced by a plane vertical wave maker.

Using the cylindrical polars (r, λ, z) as before, the hemisphere is given by $r^2 + z^2 = 1$, and the potential $\phi(r, z)\exp(-i\omega t)$ satisfies the linear conditions

$$\phi_{rr} + r^{-1}\phi_r + \phi_{zz} = 0 \quad \text{in the fluid}, \tag{4.1}$$

$$\phi + \epsilon\phi_z = 0 \quad \text{on} \quad z = 0, r > 1, \tag{4.2}$$

$$r\phi_r + z\phi_z = V_0 z \quad \text{on} \quad r^2 + z^2 = 1, \tag{4.3}$$

with ϕ finite near the circle $(r = 1, z = 0)$ where S meets the free surface. The outgoing wave condition requires that

$$\phi \sim A r^{-\frac{1}{2}} \exp\{(ir-z)/\epsilon\} \quad \text{as} \quad r \to \infty. \tag{4.4}$$

An outer approximation ϕ_0 is specified by setting $\phi_0 = 0$ on the free surface, in place of condition (4.2), and by requiring ϕ_0 to vanish at great distances from the origin. It is easy to verify that the solution is

$$\phi_0 = -\tfrac{1}{2} V_0 z (r^2 + z^2)^{-\frac{3}{2}}. \tag{4.5}$$

In particular, the solution near the rim $(r = 1, z = 0)$ is given by

$$\phi_0 \sim -\tfrac{1}{2} V_0 z \quad \text{as} \quad \delta^2 = (r-1)^2 + z^2 \to 0. \tag{4.6}$$

In terms of the inner co-ordinates

$$r = 1 + \epsilon X, \quad z = \epsilon Y, \quad \phi(r, z) = \Phi(X, Y),$$

formula (4.6) can be expressed as

$$\phi \sim \phi_0 \sim -\tfrac{1}{2} V_0 \epsilon Y \quad \text{as} \quad \epsilon \to 0, \tag{4.7}$$

which suggests an inner approximation

$$\phi \sim \Phi^{(1)} = \epsilon\Phi_0, \tag{4.8}$$

where
$$\Phi_0 \sim -\tfrac{1}{2} V_0 Y \quad \text{as} \quad R^2 = X^2 + Y^2 \to \infty \tag{4.9}$$

(together with outward-travelling waves), in order to ensure a smooth match with the outer approximation ϕ_0.

Now the boundary-value problem (4.1)–(4.4), written in terms of the local co-ordinates (X, Y), has the form

$$\Phi_{XX}+\Phi_{YY}+\Phi_X\epsilon/(1+\epsilon X) = 0 \quad (X > 0, Y > 0), \tag{4.10}$$

$$\Phi+\Phi_Y = 0 \quad (X > 0,\ Y = 0), \tag{4.11}$$

$$\Phi_X+\epsilon\{Y\Phi_Y-\tfrac{1}{2}Y^2\Phi_{XX}\}+\ldots = \epsilon^2 V_0 Y+\ldots \quad (X = 0), \tag{4.12}$$

$$\Phi \sim A\exp(iX-Y) \quad \text{as} \quad R\to\infty, \tag{4.13}$$

together with wave-free terms. The derivation of condition (4.12) from (4.3) closely follows the corresponding calculation that is discussed in detail in part 2 of this work. On substituting our first approximation $\Phi \sim \epsilon\Phi_0$ into these equations, it is found that Φ_0 is a harmonic function of X and Y, satisfying the conditions

$$\Phi_0+\Phi_{0Y} = 0 \quad (X > 0,\ Y = 0), \qquad \Phi_{0X} = 0 \quad (X = 0,\ Y > 0), \tag{4.14}$$

and the outward wave requirement (4.13), with (4.9). Its solution is

$$\Phi_0 = -\tfrac{1}{2}V_0(Y-1),$$

and is seen to be wave-free. To estimate the wave train generated towards $X = \infty$, it is therefore necessary to consider higher order terms in the inner region. The form of the boundary condition (4.12) shows a need for a term of order ϵ^2 in the expansion for Φ, though terms of intermediate order cannot be ruled out. Thus we write

$$\Phi \sim \Phi^{(2)} = \epsilon\Phi_0+\alpha(\epsilon)\,\Phi_1+\epsilon^2\Phi_2 \quad \text{as} \quad \epsilon\to 0, \tag{4.15}$$

where the scale function $\alpha(\epsilon)$ has to be found and is such that

$$\epsilon^2 \ll \alpha \ll \epsilon \quad \text{as} \quad \epsilon\to 0.$$

On substitution into the governing equations (4.10)–(4.13), it is found that the eigenfunction Φ_1 is given by

$$\Phi_1 = A(Y-1), \tag{4.16}$$

to satisfy the homogeneous conditions (4.14). Eigenfunctions that are bigger than $O(R)$ at infinity are rejected since they would lead to contributions to the inner potential Φ greater than the leading term $\epsilon\Phi_0$ in the expansion (4.15); equivalently, they could not be matched with the outer approximation $\phi \sim \phi_0$.

The function Φ_2 of formula (4.15) is required to be harmonic, and is subject to the conditions

$$\Phi_2+\Phi_{2Y} = 0 \quad (X > 0,\ Y = 0), \tag{4.17}$$

$$\Phi_{2X} = \tfrac{3}{2}V_0 Y \quad (X = 0,\ Y > 0), \tag{4.18}$$

with an *outgoing* wave requirement at ∞. This is a straightforward wave-maker problem whose solution has been given, apart from a proportionality constant, in part 2 of this work as

$$V_0^{-1}\Phi_2 = (3/\pi)\,R(\sin\theta\log R + (\theta - \tfrac{1}{2}\pi)\cos\theta) + \tfrac{3}{4}R^2\sin 2\theta$$
$$- (3/\pi)(\log R + 1) + 3G + B(Y-1), \quad (4.19)$$

where

$$G = -i\,e^{iX-Y} - \frac{1}{\pi}\int_0^\infty \frac{t\cos Yt - \sin Yt}{1+t^2}\,e^{-Xt}\,dt. \quad (4.20)$$

In particular, the wave train associated with Φ_2 is given by

$$\Phi_2 \sim -3iV_0\exp(iX - Y). \quad (4.21)$$

Completion of the second-order estimate (4.15) requires the determination of $\alpha(\epsilon)$, A and B, by matching with the outer solution through the identity $\phi^{(1,2)} \equiv \Phi^{(2,1)}$, in the notation of parts 1 and 2. Since we are interested primarily in the radiated waves, the parameters A, B and α are of little interest, but it is recorded here that $\alpha(\epsilon) = \epsilon^2\log\epsilon$.

It follows from (4.21) and (4.15) that

$$\phi \sim -3iV_0\epsilon^2\exp\{i(r-1)/\epsilon - z/\epsilon\} \quad (\epsilon \ll r-1 \ll 1), \quad (4.22)$$

together with wave-free terms that vanish when $r-1 \gg 1$. In order to deduce the far field at greater distances, formula (4.22) is simply compared with (1.13) to show that the far-field amplitude constant A of formula (4.4) is given to this order by

$$A \sim -3iV_0\epsilon^2\exp(-i/\epsilon) \quad \text{as} \quad \epsilon \to 0. \quad (4.23)$$

5. Concluding remarks

The aim of this paper and parts 1 and 2 has been to provide a plausible method for finding the short-wave asymptotic solution for a wide class of surface wave problems. The method is not rigorous, since it assumes that the potential field has expansions of certain types in different parts of the fluid region, and that the governing equations can be solved by formally substituting the assumed expansions and equating like terms in the small parameter ϵ.

The method is very plausible, however, since it leads in principle to expansions, in terms of harmonic functions, that satisfy the given boundary conditions to any order of accuracy and which smoothly match together where the different regions overlap. Furthermore, the results obtained in several two-dimensional problems are in full agreement with earlier work, much of which has a rigorous basis.

Finally it is of interest to speculate on the possibility of extending the method of this paper to deal with theproblem of travelling waves *scattered* by a three-dimensional obstacle. In the radiation problems treated in this paper, the amplitude parameter $A_0(s;\epsilon)$ (formula (1.6)) that arises from the locally two-dimensional wave pattern near C is a slowly varying function of arc length s; this implies that F (problem (1.10)) is amenable to solution by ray-theory methods. These methods need modification, however, if A_0 varies rapidly with s, as would be the case for the scattering problem associated with a hemisphere, for example, or a similar body that intersects the free surface normally.

Some speculations regarding the solution to such a problem can be made by assuming that the surface wave term ϕ_w of the potential field can still be adequately represented by a function of the form (1.9), at distances $\gg \epsilon$ from the intersection curve C. Thus on the length scale associated with the size of C, the wave function $\phi_w = F(x,y)\exp(-z/\epsilon)$ is defined effectively at all points outside C. Near C the hemisphere is locally plane, so that the local field is that of the incident potential ϕ_i, reflected by an oblique vertical wall; thus $\phi_w \approx 2\phi_i$ or $\phi_w \approx 0$ on C, according as the point is on the 'illuminated' or 'shadow' side of S. Now the function $F(x,y)$ satisfies the Helmholtz equation (cf. equation (1.10)) and evidently corresponds to the potential of a plane acoustic wave scattered by a *cylinder* of cross-section C. In particular, this suggests the conjecture that *the ratio of scattered amplitude to incident amplitude of the surface waves is the same as the corresponding ratio of amplitudes in the two-dimensional acoustics problem of scattering by a cylinder of cross-section* C.

Such acoustics problems have been investigated by many authors. Extensions, due to Keller (1956), of the ray-theory argument predict a direct and reflected field in the 'illuminated' region, and an exponentially small field in the 'shadow' region behind C. Since our outer and inner expansions are accurate only up to terms that are algebraically small in ϵ, however, the relevance of these exponentially small shadow fields to the surface wave problem is questionable: the subtle interference properties might well be completely changed by small errors in the potential on the boundary C, and an exponentially small shadow field seems unlikely.

Different scattering problems for F would emerge for different types of obstacle S. For the circular dock, for example, the local field near C would require a solution for the problem of a wave train obliquely incident upon a semi-infinite dock. A treatment for more general geometries requires the exact solution for obliquely incident waves on a beach of slope appropriate to the local shape of S.

Note added in proof. Related methods are used in recent work by Hermans (*J. Engng Maths.* 1972 **6**, 1973 **7**). The first paper concerns mainly the heaving semi-circular cylinder, and the second paper discusses three-dimensional problems.

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